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The Saturation Class and Iterates of the Bernstein Polynomials

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1. INTRODUCTION

In [3], de Leeuw identifies the saturation class for the Bernstein polynomials. His result states that $B_n f - f = O(1/n)$ if and only if $f' \in \text{Lip } 1$ on [0, 1]. This was later refined by G. Lorentz [10] to the following.

THEOREM (1.1). *If f is a function, continuous on* [0, 1], *then*

$$|B_n(f, x) - f(x)| \leqslant \tfrac{1}{2}(M/n)\, x(1 - x),$$

for $x \in [0, 1]$, $n \geqslant 1$, *is equivalent to* $f' \in \text{Lip}_M 1$ *on* [0, 1].

The method of proof used by Lorentz has also been applied to other approximation procedures [4, 14, 15].

Our purpose in this paper is to present a new proof of this theorem as well as give several generalizations of it. The novelty in our approach lies in the use of semigroup methods similar to those developed by Butzer and Berens [1].

At first it may seem that the applicability of semigroup methods is doubtful since the Bernstein polynomials do not form a semigroup. However, we can generate in a natural way a semigroup of operators from the Bernstein polynomials. This is accomplished by iterating the Bernstein polynomials as was done in [6].

Briefly, we iterate the Bernstein polynomials in the following fashion: $B_n^{k_n}$ where k_n is a non-negative integer such that k_n/n tends to t as n tends to ∞. We prove that the limit of the sequence of operators $B_n^{k_n}$ is a semigroup of class (C_0), $\mathscr{B}_t$. The identification of the saturation class for the Bernstein polynomials is now facilitated by relating it to the saturation class of $\mathscr{B}_t$ which can be determined by semigroup methods.

2. SEMIGROUPS

We incorporate in this section the necessary results concerning semigroups of operators that will be needed in our development.

Let X be a Banach space and $\{T_t: t \geq 0\}$ a one-parameter family of bounded linear operators mapping X into itself. $\{T_t: t \geq 0\}$ is said to be a semigroup of class (C_0) on X provided that

(i) $T_0 = I$,

(ii) $T_{t+s} = T_t T_s \qquad t, s \geq 0$,

(iii) $\lim_{t\to 0^+} T_t x = x \qquad x \in X$.

The infinitesimal generator of the semigroup is defined as

$$Ax = \lim_{t\to 0^+} (T_t x - x)/t,$$

whenever this limit exists.

To state the result about $\{T_t\}$ which serves fundamentally in our investigation we need additional notation. Consider a linear map $T: D(T) \to X$, $D(T) \subset X$. The powers of T are defined recursively

$$T^0 = I, \qquad \text{and} \qquad T^r = T(T^{r-1}), \quad r \geq 1.$$

$$x \in D(T^r) \qquad \text{iff} \quad x \in D(T^{r-1}) \qquad \text{and} \qquad T^{r-1}x \in D(T).$$

Let us denote the pairing between X and its dual X^* by $\langle \cdot, \cdot \rangle$,

$$\langle F, x \rangle = F(x), \qquad F \in X^*, \quad x \in X.$$

Suppose T is a densely defined linear operator on X. The adjoint operator T^* of T is the linear operator whose domain consists of the set of all $F \in X^*$ for which there exists a $G \in X^*$ such that $\langle G, x \rangle = \langle F, Tx \rangle$, for all $x \in D(T)$; in this case we set $T^*F = G$.

THEOREM 2.1. *Let $\{T_t: t \geq 0\}$ be a semigroup of class (C_0) and r a nonnegative integer, then*

$$\lim_{t\to 0^+} \left\langle F, \frac{(T_t - I)^r x}{t^r} \right\rangle = \langle (A^*)^r F, x \rangle,$$

for $F \in D((A^*)^r)$ and any $x \in X$.

Proof. The proof proceeds by induction on r. We make use of some basic results about semigroups which can be found in [1]

$$T_t x - x = A \int_0^t T_\sigma x \, d\sigma, \tag{2.1}$$

$$\lim_{t\to 0^+} \frac{1}{t} \int_0^t T_\sigma x \, d\sigma = x, \tag{2.2}$$

$$\| T_t \| \leq M e^{\beta t}, \qquad M, \beta \text{ constants.} \tag{2.3}$$

Using (2.1) and (2.2) we obtain for $F \in D(A^*)$

$$\lim_{t\to 0^+}\left\langle F, \frac{T_t x - x}{t}\right\rangle = \lim_{t\to 0^+}\left\langle F, A\frac{1}{t}\int_0^t T_\sigma x\, d\sigma\right\rangle$$
$$= \lim_{t\to 0^+}\left\langle A^*F, \frac{1}{t}\int_0^t T_\sigma x\, d\sigma\right\rangle$$
$$= \langle A^*F, x\rangle,$$

(2.3) yields the inequality

$$\left|\left\langle F, \frac{T_t x - x}{t}\right\rangle\right| \leqslant M \| A^*F\| \| x \| e^{|\beta| t}.$$

Thus, we have proved for the case $r = 1$,

$$\lim_{t\to 0^+}\left\langle F, \frac{(T_t - I)^r}{t^r} x\right\rangle = \langle (A^*)^r F, x\rangle, \qquad F \in D((A^*)^r), \quad x \in X, \quad (2.4)$$

and

$$\left|\left\langle F, \frac{(T_t - I)^r}{t^r} x\right\rangle\right| \leqslant M^r e^{r|\beta| t} \|(A^*)^r F\| \| x \|. \qquad (2.5)$$

We will prove that (2.4) and (2.5) hold for all r by induction. Suppose (2.4) and (2.5) are correct for some $k > 1$. Let $F \in D((A^*)^{k+1})$. Then $A^*F \in D((A^*)^k)$ and so referring to (2.1) we have

$$\left\langle F, \frac{(T_t - I)^{k+1} x}{t^{k+1}}\right\rangle = \left\langle A^*F, \frac{1}{t}\int_0^t T_\sigma \frac{(T_t - I)^k x}{t^k}\, d\sigma\right\rangle$$
$$= \frac{1}{t}\int_0^t \left\langle A^*F, \frac{(T_t - I)^k}{t^k} T_\sigma x\right\rangle d\sigma.$$

Invoking the induction hypothesis it follows that

$$\left|\left\langle F, \frac{(T_t - I)^{k+1}}{t^{k+1}} x\right\rangle\right| \leqslant M^k \|(A^*)^{k+1} F\| e^{k|\beta| t} \frac{1}{t}\int_0^t \| T_\sigma x\| \, d\sigma$$
$$\leqslant M^{k+1} \|(A^*)^{k+1} F\| e^{(k+1)|\beta| t} \| x \|,$$

and so (2.5) is validated for $k + 1$.

To advance induction hypothesis for (2.4) we write

$$\left\langle F, \frac{(T_t - I)^{k+1} x}{t^{k+1}}\right\rangle$$
$$= \frac{1}{t}\int_0^t \left\langle A^*F, \frac{(T_t - I)^k}{t^k}(T_\sigma x - x)\right\rangle d\sigma + \left\langle A^*F, \frac{(T_t - I)^k x}{t^k}\right\rangle.$$

The second term converges to $\langle (A^*)^{k+1}F, x\rangle$ as $t \to 0^+$ by virtue of the induction hypothesis. The first term approaches zero since

$$\left| \frac{1}{t}\int_0^t \left\langle A^*F, \frac{(T_t - I)^k}{t^k}(T_\sigma x - x)\right\rangle d\sigma \right| \leqslant \|(A^*)^{k-1}F\| M^k e^{k\beta t} \frac{1}{t}\int_0^t \|T_\sigma x - x\| d\sigma.$$

Therefore, (2.4) is also established for $k+1$ and the proof is complete.

For the application of Theorem 2.1 it is necessary to be able to identify A^*. This is made easier by introducing the concept of smooth subspace.

DEFINITION 2.1 [2]. Let X be a Banach space. A dense linear subspace of X will be called a smooth subspace of the semigroup $\{T_t; t \geqslant 0\}$ provided that

$$T_t(M) \subseteq M \qquad \text{for all} \quad t \geqslant 0,$$

and $M \subset D(A)$.

The following theorem due to de Leeuw [2] shows that the notion of smooth subspace is a useful one.

THEOREM 2.2. *Let M be a smooth subspace of the semigroup of class (C_0), $\{T_t: t \geqslant 0\}$. Then*

$$A^* = (A \mid M)^*,$$

where $A \mid M$ denotes the restriction of A to M.

3. BERNSTEIN POLYNOMIALS

As usual $C[0, 1]$ will denote the space of continuous real-valued functions defined on $[0, 1]$ normed with the sup-norm which we denote by $\|\cdot\|$.

The nth Bernstein polynomial of f is defined as

$$B_n(f(s), x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1 - x)^{n-k}.$$

We note some properties of B_n. These can all be found in [9]. First of all, they are positive linear operators on $C[0, 1]$, which leave invariant any linear function. Hence, $\| B_n \| = 1$.

For $k \leqslant n$, B_n maps the set of polynomials of degree k into itself. In particular, it can be verified that

$$B_n(s(1 - s), x) = (1 - (1/n))\, x(1 - x), \tag{3.1}$$

$$B_n((x - s)^2, x) = (1/n)\, x(1 - x). \tag{3.2}$$

Consequently, $\lim_{n\to\infty} B_n(f, x) = f(x)$ uniformly for $x \in [0, 1]$, whenever $f \in C[0, 1]$.

Another fact important in the study of the Bernstein polynomials is the inequality

$$0 \leqslant T_{ns}(x) \leqslant An^{[s/2]}, \qquad 0 \leqslant x \leqslant 1, \tag{3.3}$$

where A is a constant depending only on s and

$$T_{ns}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} (k - nx)^s.$$

Finally, we should like to note the well known asymptotic formula due to Voronowskaja.

$$\lim_{n\to\infty} n(B_n(f, x) - f(x)) = \tfrac{1}{2}x(1-x)f''(x).$$

This is valid provided f has a second derivative at $x \in [0, 1]$. Moreover, the convergence is uniform in x, whenever f'' is continuous on $[0, 1]$.

We now turn our attention to the iterates of B_n. Let $A_n = n(B_n - I)$, then

$$B_n^{k_n} = \left(I + \frac{A_n}{n}\right)^{k_n} = f_n(A_n),$$

where

$$f_n(z) = (1 + (z/n))^{k_n}.$$

Noting that $\lim_{n\to\infty} f_n(z) = e^{tz}$ uniformly on compact subsets of the plane, whenever $k_n/n \to t$ and

$$\lim_{n\to\infty} A_n(f, x) = A(f, x) = \tfrac{1}{2}x(1-x)f''(x), \qquad f'' \in C[0, 1],$$

we might expect that $\lim_{n\to\infty} B_n^{k_n} = e^{tA}$. This is, in fact, the case and can be proven by restricting the Bernstein polynomials to the subspace of polynomials of a fixed degree. In this way, the operators become ordinary matrices and convergence then follows immediately. The convergence persists on all of $C[0, 1]$ since polynomials are dense and B_n has norm 1.

THEOREM 3.1. *There exists a semigroup $\{\mathscr{B}_t : t \geqslant 0\}$ of class (C_0) on $C[0, 1]$ such that*

$$\lim_{n\to\infty} B_n^{k_n} f = \mathscr{B}_t f, \qquad \textit{whenever} \quad \lim_{n\to\infty} (k_n/n) = t,$$

for all $t \geqslant 0$ and $f \in C[0, 1]$. $\{\mathscr{B}_t : t \geqslant 0\}$ is a positive contraction semigroup. Its infinitesimal generator A has the property that whenever φ is an infinitely *differentiable function having support interior to* [0, 1] *and* $F_\varphi(g) = \int_0^1 \varphi(x)\, g(x)\, dx$ *then* $F_\varphi \in D(A^*)$ *and*

$$A^* F_\varphi = F_{A^*\varphi},$$

where $(A^*\varphi)(x) = (\frac{1}{2}x(1-x)\,\varphi(x))''$. *Moreover, the following inequality is valid*

$$\|(1/t)(\mathscr{B}_t f - f) - g\| \leqslant \varliminf_{n\to\infty} \|n(B_n f - f) - g\| + \left\|(1/t)\int_0^t B_\sigma g\, d\sigma - g\right\|,$$

for any $f, g \in C[0, 1]$ *and* $t > 0$.

Proof. Let $0 \leqslant l \leqslant n$. For convenience we do not distinguish between B_n and its restriction to the linear space of polynomials of degree less than or equal to l which we denote by P_l. B_n maps P_l isomorphically into itself. Moreover, if $p \in P_l$, we have

$$\lim_{n\to\infty} n(B_n p - p) = Ap. \tag{3.4}$$

Hence,

$$\lim_{n\to\infty} B_n^{k_n} p = \lim_{n\to\infty} f_n(A_n)p = e^{tA}p.$$

The last equality follows from (3.4) and

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} (1 + (z/n))^{k_n} = e^{tz}.$$

Thus, we have established the convergence of the iterates for polynomials. However $\| B_n{}^k \| = 1$ for all nonnegative integers k and n. Therefore, the limit exists for all $f \in C[0, 1]$.

$$\lim_{n\to\infty} B_n^{k_n} f = \mathscr{B}_t f.$$

On polynomials we have the explicit representation

$$\mathscr{B}_t p = e^{tA}p,$$

for $\mathscr{B}_t$. This is sufficient to verify that $\mathscr{B}_t$ is a semigroup of class (C_0) if one keeps in mind that $\| \mathscr{B}_t \| = 1$. Moreover, we see that every polynomial p is in $D(A)$ and

$$A(p, x) = \tfrac{1}{2}x(1-x)(d^2/dx^2)\, p(x).$$

An integration by parts can be utilized to verify that identity

$$\langle F_\varphi, Ap\rangle = \langle F_{A^*\varphi}, p\rangle,$$

whenever p is a polynomial. But polynomials form a smooth subspace of $\mathscr{B}_t$ and so according to Theorem 2.2, we infer that $F_\varphi \in D(A^*)$ and

$$A^*F_\varphi = F_{A^*\varphi}.$$

The inequality is now the only remaining assertion to be proved. Let k be a positive integer then

$$B_n^k f - f = \sum_{j=0}^{k-1} B_n^j(B_n f - f),$$

$$\begin{aligned}\frac{1}{t}(B_n^k f - f) - g &= \frac{1}{t}\sum_{j=0}^{k-1} B_n^j(B_n f - f) - g \\ &= \frac{1}{nt}\sum_{j=0}^{k-1} B_n^j[n(B_n f - f) - g] + \frac{1}{nt}\sum_{j=0}^{k-1} B_n^j g - g.\end{aligned}$$

Therefore,

$$\left\| \frac{1}{t}(B_n^k f - f) - g \right\| \leqslant \frac{k}{nt} \| n(B_n f - f) - g \| + \left\| \frac{1}{nt}\sum_{j=0}^{k-1} B_n^j g - g \right\|.$$

We will be finished provided that we prove

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{[nt]-1} B_n^j g = \int_0^t \mathscr{B}_\sigma g \, d\sigma, \tag{3.5}$$

for $g \in C[0, 1]$. Again, since $\|(1/n)\sum_{j=0}^{[nt]-1} B_n^j g\| \leqslant t \| g \|$ it will suffice to prove (3.5) in the case of polynomials. Let p be a polynomial of degree $\leqslant l$. Then

$$\frac{1}{n}\sum_{j=0}^{[nt]-1} B_n^j p = F_n(A_n)p,$$

where $F_n(z)$ is defined as

$$F_n(z) = \frac{1}{n}\sum_{j=0}^{[nt]-1}\left(1 + \frac{z}{n}\right)^j = \left[\left(1 + \frac{z}{n}\right)^{[nt]} - 1\right]\Big/ z,$$

as n tends to ∞, $F_n(z)$ approaches $(e^{tz} - 1)/z$ uniformly on compact subsets of the plane. We, therefore, can infer the validity of (3.5).

Remark. Trotter studied in [16] the question of convergence of sequences of semigroup on B-spaces. Theorem 3.1 is partially subsumed by his more general results. However, in the case under consideration the proof we presented is direct and elementary. Another elementary proof of the convergence of the iterates was given in Kelisky and Rivlin [6].

THEOREM 3.2. *Let f be a real-valued function defined on* $[0, 1]$ *then* (i) *through* (iii) *are equivalent statements.*

(i) $|f'(x) - f'(y)| \leq M\,|x - y|,\ x, y \in [0, 1],\ (f' \in \mathrm{Lip}_M\, 1)$.

(ii) $|B_n(f, x) - f(x)| \leq (M/2n)\, x(1 - x),\ n \geq 1,\ x \in [0, 1]$.

(iii) $|\mathscr{B}_t(f, x) - f(x)| \leq (Mt/2)\, x(1 - x),\ t \geq 0,\ x \in [0, 1]$.

Moreover,

(iv) *if* $$\lim_{n\to\infty} \| n(B_n f - f) - g \| = 0,$$

where $f, g \in C[0, 1]$ *then* $f'' \in C(0, 1)$ *and* $\frac{1}{2}x(1 - x) f''(x) = g(x),\ 0 < x < 1$.

Proof. (i) $\Rightarrow$ (ii). We follow the analysis in [10]

$$f(x) - f(s) = \int_s^x f'(t)\, dt = f'(x)(x - s) - \int_s^x (t - s)\, df'(t)$$

$$|f(x) - f(s) - f'(x)(x - s)| \leq M \left| \int_s^x (t - s)\, dt \right| = \frac{M}{2}(x - s)^2.$$

Since B_n is a positive operator we can operate on the variable s in the previous inequality and obtain

$$|f(x) - B_n(f, s) - f'(x)(x - B_n(s, x))| \leq (M/2)\, B_n((x - s)^2, x)$$
$$|B_n(f, x) - f(x)| \leq (M/2)\, B_n((s - x)^2, x).$$

(ii) now follows from (3.2).

(ii) $\Rightarrow$ (iii)

$$B_n^k(f, x) - f(x) = \sum_{l=0}^{k-1} B_n^l(B_n f - f, x).$$

Thus,

$$|B_n^k(f, x) - f(x)| \leq \frac{M}{2n} \sum_{l=0}^{k-1} B_n(s(1 - s), x).$$

Making use of (3.1) the right side simplifies to

$$(M/2)[1 - (1 - 1/n)^k]\, x(1 - x).$$

Therefore, passing to the limit

$$|\mathscr{B}_t(f, x) - f(x)| \leq (M/2)(1 - e^{-t})\, x(1 - x) \leq (M/2)\, tx(1 - x).$$

(iii) $\Rightarrow$ (i). Let $\varphi \in C_0^\infty(a, b)$, an infinitely differentiable function whose support is interior to $[a, b]$. Set

$$\lambda_n(x) = n \int_a^x \frac{\mathscr{B}_{1/n}(f, \sigma) - f(\sigma)}{\frac{1}{2}\sigma(1 - \sigma)} d\sigma, \qquad x \in [a, b].$$

Then $| \lambda_n(x) - \lambda_n(x)| \leqslant M \mid x - y \mid$, for $x, y \in [a, b]$ and $\lambda_n(a) = 0$. Hence, by the Helly selection theorem there exists a sequence $n_k \to \infty$ such that

$$\lim_{k \to \infty} \lambda_{n_k}(x) = \lambda(x), \qquad \text{all} \quad x \in [a, b],$$

and

$$\lim_{k \to \infty} \int_a^b g(x)\, d\lambda_{n_k}(x) = \int_a^b g(x)\, d\lambda(x), \qquad \text{for} \quad g \in C[a, b].$$

Observe also that $\lambda(x)$ is in $\text{Lip}_M\, 1$ on $[a, b]$.

Employing Theorem 2.1 and Theorem 3.1, we obtain

$$\lim_{k \to \infty} \int_a^b \tfrac{1}{2}x(1 - x)\, \varphi(x)\, d\lambda_{n_k}(x) = \int_0^1 (\tfrac{1}{2}x(1 - x)\, \varphi(x))'' f(x)\, dx.$$

Thus,

$$\int_a^b \tfrac{1}{2}x(1 - x)\, \varphi(x)\, d\lambda(x) = \int_a^b (\tfrac{1}{2}x(1 - x)\, \varphi(x))'' f(x)\, dx.$$

Integrating by parts twice

$$\int_a^b \varphi''(x) \left[f(x) - \int_a^x \lambda(t)\, dt\right] dx = 0, \qquad \text{all} \quad \varphi \in C_0^\infty(a, b).$$

We deduce as in Lorentz [10] that $f(x) - \int_a^x \lambda(t)\, dt$ is a linear function. This establishes (iii). The proof of (iv) proceeds similarly from the inequality proven in Theorem 3.1.

This method of iteration can be utilized to prove certain other known properties of the Bernstein polynomials. For instance, it was first proved in [3] that $\| B_n f - f \| = o(1/n)$ if and only if f is a linear function on [0, 1].

Let us show how this can be easily proven by iterating the Bernstein polynomials. If we assume that $\| B_n f - f \| = o(1/n)$, then

$$\| B_n^k f - f \| \leqslant (k/n)\, \epsilon_n\,,$$

where ϵ_n tends to zero as n tends to ∞.

Choose a sequence $\{k_n\}$ of integers so that $\lim_{n\to\infty} (k_n/n) = \infty$ and $\lim_{n\to\infty} (k_n/n)\, \epsilon_n = 0$. Then $\lim_{n\to\infty} B_n^{k_n} f = f$. However, Kelisky and Rivlin [6] proved that $\lim_{n\to\infty} B_n^{k_n} f = B_1 f$, for all $f \in C[0, 1]$. Therefore, $B_1 f = f$ which is the desired conclusion. We remark that a similar proof gives the "little o" theorem for the Bernstein polynomials on the N-cube or N-simplex [13].

As another application of the method of iteration we prove that

$$B_n(f, x) \geqslant f(x), \qquad n \geqslant 1, \qquad x \in [0, 1], \tag{3.6}$$

for $f \in C[0, 1]$ iff f is convex on $[0, 1]$. One way is immediate. In fact, if f is convex then (3.6) follows from Jensen's inequality. Conversely, if (3.6) is valid then upon iteration, we obtain

$$\mathscr{B}_t(f, x) \geqslant f(x), \qquad t \geqslant 0, \qquad x \in [0, 1].$$

Let ϕ be a nonnegative function in $C_0^\infty(0, 1)$, then

$$0 \leqslant \lim_{t\to 0^+} \frac{1}{t} \int_0^1 \frac{\phi(x)}{\frac{1}{2}x(1-x)} [\mathscr{B}_t(f, x) - f(x)]\, dx = \int_0^1 \phi''(x) f(x)\, dx.$$

Thus, f is convex.

4. Generalizations

We give several other examples for which our method is applicable.

Example 1. From [5] the following generalization of the Bernstein polynomials is suggested. Let F be a probability generating function

$$F(x) = \sum_{n=0}^{\infty} a_n s^n, \qquad a_n \geqslant 0 \qquad \text{and} \qquad \sum_{n=0}^{\infty} a_n = 1.$$

Define

$$P_{ij}^n = \frac{\text{coeff } s^j t^{n-j} \text{ in } (F(s))^i (F(t))^{n-i}}{\text{coeff } \omega^n \text{ in } (F(\omega))^n}. \tag{4.1}$$

For instance, if $F(s) = e^{s-1}$ it is readily verified that

$$P_{ij}^n = \binom{n}{j} (i/n)^j (1 - i/n)^{n-j}. \tag{4.2}$$

Define P_n: $C[0, 1] \to C(X_n)$, $X_n = \{0/n, 1/n, \ldots, n/n\}$

$$P_n(f, i/n) = \sum_{j=0}^{n} P_{ij}^n f(j/n). \tag{4.3}$$

For $h \in C(X_n)$ we set $\| h \|_n = \max_{0\leqslant i\leqslant n} | h(i/n)|$.

In the case of $F(s) = e^{s-1}$ we see that $P_n(f, i/n)$ agrees with the Bernstein polynomials at i/n

$$P_n(f, i/n) = B_n(f, i/n). \tag{4.4}$$

(4.4) suggests interpolating the values $\{P_n(f, i/n)\}_{i=0}$ with a polynomial of degree n. If we call the operator thus formed $R_n(f, x)$, then (4.4) insures that these certainly generalize the Bernstein polynomials. But, obviously, we cannot expect that $R_n(f, x)$ converges to $f(x)$ for every continuous function as the case $F(s) = s$ attests. In this case $R_n(f, x) = L_n(f, x)$, where $L_n(f, x)$ is the Lagrange polynomial of degree n which interpolates f on X_n. A possible alternative is to use a different polynomial basis. For instance

$$T_n(f, x) = \sum_{k=0}^{n} P_n(f, k/n) \binom{n}{k} x^k(1 - x)^{n-k}. \tag{4.5}$$

This sequence of operators form a proper generalization of the Bernstein polynomials in the sense that $T_n f$ converges to f for all continuous functions. Nevertheless, it is not known under what conditions on the generating function $F(s)$ would insure that $R_n f$ also has this property. The saturation class of the family of operators $\{T_n\}$ is identified in Theorem 4.2. Define

$$\lambda_2^n = \frac{\text{coeff of } z^{n-2} \text{ in } f^{(n-2)}(z)(f'(z))^2}{\text{coeff of } z^n \text{ in } (f'(z))^n}$$

We can easily verify that

$$\| P_n f - f \|_n \leqslant \tfrac{3}{2}\omega(f, (1 - \lambda_2^n)^{1/2}),$$

where $\omega(f, \delta)$ is the modulus of continuity of f.

In [17, 13], it was shown that whenever $f''(1) < \infty$

$$\lim_{n\to\infty} n(1 - \lambda_2^n) = \gamma \geqslant 0.$$

Consequently, we see that the rate of approximation of a continuous function f by either $B_n f$ or $P_n f$ is the same. Also, in [17] and [13], it was shown that

$$\lim_{n\to\infty} \| P_n^{[nt]} f - \mathscr{B}_{\gamma t} f \|_n = 0.$$

Hence, proceeding as in Theorem 3.2 the following can be proved.

THEOREM 4.1. *Let $f \in C[0, 1]$ and suppose $\gamma > 0$, then* (i) $f' \in \mathrm{Lip}_M 1$ *on* $[0, 1]$ *iff*

$$| P_n(f, i/n) - f(i/n)| \leqslant \tfrac{1}{2}M(1 - \lambda_2^n)(i/n)(1 - i/n), \qquad 0 \leqslant i \leqslant n.$$

(ii) *If* $\underline{\lim}_{n\to\infty} \| n(P_n f - f) - g \|_n = 0$ *and* $g \in C[0, 1]$ *then*

$$f'' \in C(0, 1) \quad \text{and} \quad (\gamma/2)\, x(1 - x) f''(x) = g(x), \quad 0 < x < 1.$$

(iii) *If* $\| P_n f - f \|_n = o(1 - \lambda_2{}^n)$ *then* f *is linear on* $[0, 1]$.

(iv) f *is convex on* $[0, 1]$ *iff* $P_n(f, i/n) \geq f(i/n)$, $0 \leq i \leq n$.

Note that from (4.4), Theorem 4.1 gives a stronger result for the Bernstein polynomials than either Theorem 3.2 or (3.6).

EXAMPLE 2. Let $\{T_n : n \geq 0\}$ be a sequence of nonnegative linear operators mapping $C[0, 1]$ into itself. Suppose $T_n \neq I$ or B_1 for $n \geq 2$ and

(i) T_n preserves linear functions and

(ii) T_n takes quadratics into quadratic.

Such a sequence of operators necessarily satisfy

$$\begin{aligned} T_n(f, 0) &= f(0), \\ T_n(f, 1) &= f(1). \end{aligned} \tag{4.8}$$

Moreover, from (ii) we see there exists a constant λ_n, $0 < \lambda_n < 1$, $(n \geq 2)$ exists such that

$$T_n(s(1 - s), x) = \lambda_n x(1 - x). \tag{4.9}$$

In the case that T_n is the nth Bernstein polynomial $\lambda_n = 1 - 1/n$. Furthermore, if we assume that

$$\lim_{n\to\infty} \lambda_n = 1 \quad \text{and} \quad \lim_{n\to\infty} (1 - \lambda_n)^{-1}\, T_n((x - s)^4, x) = 0$$

uniformly in x on $[0, 1]$ then by using Taylor's theorem it can be shown that for any l there exists a sequence $\epsilon_n \to 0$ such that for every polynomial $p \in P_l$

$$\begin{aligned} \|(1 - \lambda_n)^{-1} [T_n p - p] - Ap \| &\leq \epsilon_n \| p \|, \\ A(p, x) &= \tfrac{1}{2} x(1 - x)\, p''(x). \end{aligned} \tag{4.10}$$

The constant C depends only on the degree of p. In particular, (4.10) holds for $\mathscr{B}_{(1-\lambda_n)}$, and so we have for some $\bar{\epsilon}_n \to 0$

$$\| T_n p - \mathscr{B}_{(1-\lambda_n)} p \| \leq \bar{\epsilon}_n (1 - \lambda_n)^1 \| p \|, \qquad p \in P_l .$$

Since polynomials are invariant under $\mathscr{B}_{(1-\lambda_n)}$, it follows that

$$\| T_n{}^k p - \mathscr{B}_{k(1-\lambda_n)} p \| \leq \bar{\epsilon}_n k (1 - \lambda_n)^1 \| p \| .$$

Therefore, for all $f \in C[0,1]$

$$\lim_{n\to\infty} T_n^{[t(1-\lambda_n)^{-1}]} f = \mathscr{B}_t f.$$

Proceeding as in Theorem 3.2 we can show the following.

THEOREM 4.2. *Let $f \in C[0, 1]$ then*

(i) $|T_n(f, x) - f(x)| \leqslant (M/2)(1 - \lambda_n)\, x(1 - x)$ *if and only if* $f' \in \mathrm{Lip}_M\, 1$.

(ii) $lim_{n\to\infty} \|(1 - \lambda_n)^{-1}(T_nf - f) - g\| = 0$, $g \in C[0, 1]$, *implies*

$$f'' \in C(0, 1) \qquad \text{and} \qquad \tfrac{1}{2}x(1 - x)f''(x) = g(x) \text{ on } [0, 1].$$

(iii) *If* $\| T_nf - f\| = o(1 - \lambda_n)$, *then f is linear on* $[0, 1]$.

(iv) $T_n(f, x) \geqslant f(x)$ *if and only if f is convex on* $[0, 1]$.

Let us now return to the Bernstein polynomials. We define for k, a nonnegative integer, the sequence of linear operators

$$U_{n,k} = I - (I - B_n)^k = -\sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} B_n^{\nu}.$$

If $\omega(f, \delta)$ is the modulus of continuity of f then one can easily verify

$$| U_{n,k}(f, x) - f(x)| \leqslant (3/2)(2^k - 1)\, \omega(f, 1/\sqrt{n}).$$

It appears then that $U_{n,k}f$ provides no better an approximation to f than B_nf itself. However, unlike B_n, we can improve the order of approximation beyond $O(1/n)$ for sufficiently smooth functions.

THEOREM 4.4. *Suppose k is a nonnegative integer. If $f, f', \ldots, f^{(2k+1)}$ are in* $C[0, 1]$ *and* $f^{(2k+1)} \in \mathrm{Lip}\ 1$ *on* $[0, 1]$, *then*

$$| U_{n,k+1}(f, x) - f(x)| = O\left(\frac{1}{n^{k+1}}\right)$$

uniformly for $x \in [0, 1]$.

Proof. Define

$$\|f\|_k = \max_{0\leqslant j\leqslant 2k+1} \{\|f^j\|, M[f^{(2k+1)}]\},$$

where $M[f^{(2k+1)}]$ is the Lipschitz constant for $f^{(2k+1)}$. We will prove by induction that

$$| U_{n,k+1}(f, x) - f(x)| \leqslant \frac{C_k \|f\|_k}{n^{k+1}},$$

C_k is a constant independent of n, f and $x \in [0, 1]$. The case $k = 0$ is covered in Theorem 3.1. Suppose the theorem is correct for all $j < k$ and let f satisfy the hypothesis of the theorem, then

$$f(x) - \sum_{l=0}^{2k+1} \frac{f^{(l)}(x_0)}{l!}(x - x_0)^l = \frac{1}{(2k + 1)!}\int_{x_0}^{x} (x - \sigma)^{2k+1}\, df^{(2k+1)}(\sigma).$$

Using the positivity of B_n we have

$$\left| B_n(f, x) - \sum_{l=0}^{2k+1} \frac{f^{(l)}(x)}{n^l l!} T_{n,l}(x) \right| \leqslant \frac{\|f\|_k}{n^{2k+2}(2k+1)!} T_{n,2k+2}(x).$$

According to (3.3) we can rewrite this inequality as

$$(B_n - I)(f, x) = \sum_{l=2}^{2k} \frac{f^{(l)}(x)}{l!\, n^l} T_{n,l}(x) + g_n(x),$$

where

$$|g_n(x)| \leqslant \frac{a_k \|f\|_k}{n^{k+1}}.$$

The functions

$$F_{n,l}(x) = \frac{f^{(l)}(x)\, T_{n,l}(x)}{l!\, n^{[l/2]}}, \qquad l = 2,..., 2k,$$

satisfy the hypothesis of the theorem for the integer $k_l = [k - l/2]$. Since $k_l < k$ for $l = 2,..., 2k$, the induction hypothesis implies

$$|(B_n - I)^{k_l+1} (F_{n,l} x)| \leqslant \frac{d_k}{n^{k_l+1}} \| F_{n,l} \|_{k_l}.$$

But $T_{n,l}/l!\, n^{[l/2]}$ is a polynomial of degree l which is uniformly bounded in n, and, moreover, $\|(B_n - I)^j\| \leqslant 2^j$ for $j \geqslant 0$. Therefore, we obtain

$$|(B_n - I)^k (F_{n,l}, x)| \leqslant \frac{b_k}{n^{k_l+1}} \|f\|_k, \qquad l = 2,..., 2k.$$

Combining our inequalities yields

$$\begin{aligned} |U_{n,k+1}(f, x) - f(x)| &= \left| (B_n - I)^k \left(\sum_{l=2}^{2k} \frac{F_{n,l} n^{[l/2]}}{n^l} + g_n, x \right) \right| \\ &\leqslant \left\{ b_k \sum_{l=2}^{2k} \frac{n^{[l/2]}}{n^{k_l+1} n^l} + \frac{2^k a_k}{n^{k+1}} \right\} \|f\|_k \\ &= \frac{2^k a_k + b_k(2k-1)}{n^{k+1}} \|f\|_k. \end{aligned}$$

Therefore, the theorem is valid for k and the induction is complete.

THEOREM 4.5. *Let $f \in C[0, 1]$ and k be a nonnegative integer. If*

$$|U_{n,k+1}(f, x) - f(x)| \leqslant \frac{Mx(1-x)}{2n^{k+1}} + o\left(\frac{1}{n^{k+1}}\right),$$

uniformly for $x \in [0, 1]$. *Then* $f^{(i)} \in C(0, 1)$, $i = 0, 1, \ldots, 2k + 1$ *and* $A^k f$, *the* kth *power of the operator* $(Af)(x) = \frac{1}{2}x(1 - x)f''(x)$, *has a continuous extension to* $[0, 1]$ *whose derivative is in* $\mathrm{Lip}_M\, 1$.

Proof. Set $S_{j,n} = \sum_{l=0}^{j-1} B_n{}^l$. Then

$$B_n{}^j - I = S_{j,n}(B_n - I), \qquad \text{and} \qquad S_{j,n}B_n = B_n S_{j,n}. \tag{4.11}$$

Moreover, $S_{n,j}$ is a positive operator with $\| S_{n,j} \| = j$, and (4.11) implies

$$(B_n{}^j - I)^{k+1} = S_{j,n}^{k+1}(B_n - I)^{k+1} = S_{j,n}^{k+1}(U_{n,k+1} - I). \tag{4.12}$$

If f satisfies the hypothesis of the theorem then

$$|(B_n{}^j - I)^{k+1}(f, x)| \leqslant \frac{M}{2n^{k+1}} S_{j,n}^{k+1}(s(1 - s), x) + j^{k+1} o\left(\frac{1}{n^{k+1}}\right),$$

$$S_{j,n}^{k+1}(s(1 - s), x) = n^{k+1}[1 - (1 - \lambda_n)^j]^{k+1}\, x(1 - x).$$

Letting $j = [nt]$ yields in the limit

$$|(\mathscr{B}_t - I)^{k+1}(f, x)| \leqslant (M/2)\, t^{k+1} x(1 - x).$$

Suppose $\phi \in C_0{}^\infty(a, b)$, $0 < a < b < 1$ and $F_\phi(g) = \int_0^1 \phi(x)\, g(x)\, dx$.

Theorem 3.1 implies $F_\varphi \in D((A^*)^{k+1})$ and $(A^*)^{k+1} F_\varphi = F_{(A^*)^{k+1}\varphi}$. Define

$$\lambda_t(x) = \frac{1}{t^{k+1}} \int_a^x \frac{(\mathscr{B}_t - I)^{k+1}(f, \sigma)}{\frac{1}{2}\sigma(1 - \sigma)}\, d\sigma, \qquad a \leqslant x \leqslant b.$$

Making use of Theorem 2.1 and the Helly selection theorem, as in Theorem 3.1, we are assured that there exists a $\lambda \in \mathrm{Lip}_M\, 1$ on $[a, b]$ such that

$$\int_a^b \tfrac{1}{2}x(1 - x)\, \varphi(x)\, d\lambda(x) = \int_a^b ((A^*)^{k+1} \varphi)(x)\, f(x)\, dx,$$

for all $\varphi \in C_0{}^\infty(a, b)$. Integration by parts yields a function $G \in C[a, b]$ such that $G^{(i)} \in C[a, b]$, $i = 0, 1, \ldots, 2k + 1$,

$$(A^k G)' = \lambda \quad \text{on} \quad [a, b]$$

and

$$\int_a^b (A^*)^{k+1}\, \varphi(x)\, F(x)\, dx = 0,$$

where $F(x) = f(x) - G(x)$.

It follows by an elementary argument that $F \in C^\infty[a, b]$ and

$$(A^k F)''(x) = 0, \qquad x \in [a, b].$$

This implies $f^{(i)} \in C[a, b]$, $i = 0, 1, \ldots, 2k+1$ and $(A^k f)' \in \mathrm{Lip}_M\, 1$ on $[a, b]$. Since a and $b (a < b)$ were chosen arbitrarily from $(0, 1)$ we have $A^k f \in C(0, 1)$ and $(A^k f)' \in \mathrm{Lip}_M\, 1$ on $(0, 1)$. The desired conclusion is hereby established.

COROLLARY 4.2. *Let f be a real-valued function defined on* $[0, 1]$. *If* $| U_{n,k+1}(f, x) - f(x)| = o(1/n^{k+1})$ *uniformly on* $[0, 1]$, *then f is linear on* $[0, 1]$.

Proof.

$$\|(B_n{}^j - I)^{k+1} f\| \leqslant j^{k+1} \| U_{n,k+1} f - f \| \leqslant (j/n)^{k+1} n^{k+1} \| U_{n,k+1} f - f \|.$$

Hence, there exists a sequence $\{j_n\}$ of integers such that $j_n/n \to \infty$, $\lim_{n\to\infty} (B_n^{j_n} - I)^{k+1} f = 0$. It was previously pointed out that

$$\lim_{n\to\infty} B_n^{j_n}(f, x) = B_1(f, x),$$

whenever $(j_n/n) \to \infty$. Therefore, $(B_1 - I)^{k+1}(f, x) = 0$. Taking advantage of the relationship $B_1{}^2 = B_1$, we see that

$$(B_1 - I)^{k+1} = (-1)^k (B_1 - I).$$

Thus, $f = B_1 f$.

Remark. In light of the recent interesting work of Lorentz and Schumaker [11] we would like to comment on the range of applicability of our method.

With a few appropriate modifications, it can be extended to positive linear operators on $C[0, 1]$ which have the following asymptotic formula

$$T_n(f, x) = f(x) + \tfrac{1}{2}\sigma(x)\, D_2 D_1 f(x) + o(1/\lambda_n),$$

where $\lim_{n\to\infty} \lambda_n = \infty$, $\sigma(x) \geqslant 0$, $D_i f(x) = (d/dx)(f(x)/W_i(x))$, and $0 < a < W_i(x) < b$. We also require $\sigma(x) > 0$ in $(0, 1)$. Otherwise, λ_n may not be the order of saturation. If we denote

$$\phi_0(x) = W_0(x),$$

$$\phi_1(x) = W_0(x) \int_0^x W_1(\sigma)\, d\sigma,$$

$$\sigma(x, y) = -\int_x^y W_0(x) \int_t^x W_1(\sigma)\, d\sigma\, dt,$$

then

$$\lim_{n\to\infty} \lambda_n \| T_n \phi_i - \phi_i \| = 0, \qquad i = 0, 1,$$

$$\lim_{n\to\infty} \lambda_n T_n(\sigma(\cdot, y), y) = \sigma(y).$$

Thus, whenever the iterates of T_n converge to a semigroup (conditions on $\sigma(x)$ can be formulated insuring convergence by using the results in [7, 12]) then

$$| T_n(f, x) - f(x)| \leqslant (1/\lambda_n)(M\sigma(x) + o(1)),$$

if and only if $f \in \mathrm{Lip}_M\ 1$.

We would also like to note that in [13] several other approximation procedures are considered from the point of view of iteration. These include convolution approximation procedures for 2π-periodic functions and certain "expected value" approximation procedures on the infinite and semi-infinite line.

Remark. It has been brought to the author's attention that there is some duplication between this paper and work done independently by R. Schnable, Zum Globalen Saturationsproblem der Folge der Bernsteinoperatoren, *Acta. Sci. Math.* (Szeged) (to appear). He used similar methods to prove Theorem 3.2.

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